Contributions of contact nonlinearities to wheel/rail noise generation

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Summary
A model for calculation of generation and radiation of railway noise is presented. Thanks to a Green’s function formulation, in the time as well as in the frequency domain, not only roughness excitation, but also parametric excitation (due to varying rail receptance along the track) and excitation due to nonlinear effects in the rail/wheel contact region are included in the description. Another advantage of the model is that vibration propagation and decay along the track is an inherent property of the Green’s function of the periodically supported rail. The advantages of the model is demonstrated with a calculation example of rail/wheel contact force and noise radiation from the rail for a typical sleeper-supported ballast track. Noise radiation peaks around 1000 Hz. For higher frequencies, parametric excitation and nonlinear effects may contribute significantly to total railway noise radiation.

1. Theory
1.1 Rail vibration and radiation
1.1.1 Point force excitation
Sound power radiation from a vibration surface
\[ W = \frac{1}{2} \rho_0 c_0 \sigma S \langle v^2 \rangle \]
where \( \rho_0 \) is density of air, \( c_0 \) air speed, \( \sigma \) radiation efficiency, and \( S \) the area of the radiating surface. \( \langle v^2 \rangle \) is mean square of the vibrating velocity of the radiating surface.

Consider a unit point force excited infinite rail. The vibration amplitude decays away from the force point. Total radiation from the infinite rail
\[ W = \frac{1}{2} \rho_0 c_0 \sigma w \int_{-\infty}^{\infty} |v(x)|^2 dx \]
where \( w \) is the width of the rail, which may be rewritten
\[ W = \frac{1}{2} \rho_0 c_0 \sigma \omega^2 \int_{-\infty}^{\infty} |G(x|x_0)|^2 dx \]
where \( \omega \) is the circular frequency. \( G(x|x_0) \) is the Green’s function of the periodically, sleeper-supported, infinite rail [1,2,3]. The force acts in the point \( x_0 \), and \( x \) is the response position. The decay of rail vibrations along the rail in the \( x \)-direction is automatically taken account of by the propagation coefficients in the Green’s function.

1.1.2 Rail/wheel contact force excitation
Now, the contact force \( F(x) \) between rail and wheel excites the rail. The expression for sound radiation then becomes [1,3]
\[ W = \frac{1}{2} \rho_0 c_0 \sigma \omega^2 \int_{-\infty}^{\infty} |F(x)G(x|x_0)|^2 dx \]
If the contact force spectrum is independent of the wheel position in a sleeper span, the force may be taken outside of the integral,
\[ W = \frac{1}{2} \rho_0 c_0 \sigma \omega^2 |F|^2 \int_{-\infty}^{\infty} |G(x|x_0)|^2 dx \]
The contact force depends on rail/wheel running surface roughness levels, geometrical filtering and compression in the contact region which are nonlinear processes, rail and wheel receptances, as well as variation in the rail receptance along the rail with wheel position due to the discrete supports. It can be calculated, according to section 1.3.1 in the time-domain with the rail/wheel simulation program [4], and the nonlinear state-dependent rolling contact model [5]. Alternatively, the linear contact model in section 1.3.2 can be used.
1.2 Rail Green’s function in the frequency domain

Green’s function $G_\omega(x,x_0)$, in the frequency domain for a periodically supported rail is in detail derived in [1,2,3,4], enabling calculation of rail response in any position to any excitation. It is used below to determine rail/wheel interaction contact force and rail vibration/radiation levels. The model accounts for the parameters defined in Figure 1 and Table 1.

1.3 Contact force calculation

1.3.1 Time domain model

A wheel with mass $M_w$ and preload $P$ rolls with forward velocity $v$ over a rail (Figure 1). The rail has mass $m_r$ per unit length, bending stiffness $EI$. In the figure, the rail symbolically has a corrugation $y_s = r_0 \sin(2\pi f_0 x)$ on the running surface. The coordinate axis for $x$ points in the forward running direction, and for $y$ upwards in the vertical direction. A perfectly smooth rail implies that $y_s = 0 \ \mu m$. The rail is periodically supported at discrete points with spacings $l$. Each support consists of a spring–mass–spring combination: $K_p$ denotes pad stiffness with loss factor $\eta_p$, $M_s$ sleeper mass and $K_b$ ballast stiffness with loss factor $\eta_b$. The compression in the wheel–rail contact region, $\delta_{\text{lin}} = \gamma_r + y_s - y_w$ is a function of vertical rail, $\gamma_r$ and wheel, $y_w$ deflections plus the deviation from a perfect running surface, $y_s$. The contact force $f$, caused by the compression, excites the wheel (upwards) and the rail (downwards). Since this is a time-domain model, rail and wheel deflections, plus the contact force are calculated at each discrete point $x_0 = n\Delta x$, where $n$ is an integer ad $\Delta x$ the space increment, with the corresponding time increment $\Delta t = \Delta x / v$.

Rail — impulse response function

The vertical rail deflection, $y_r(x_0,t_0)$ at the moving point, $x_0 = vt$, (under the wheel) is a convolution of the rail Green's function, $g(x,x_0; t,t_0)$, and the contact force $f(x,t)$.

$$y_r(x_0,t_0) = \int \int g(x,x_0; t,t_0)f(x,t)dxdt$$

After discretisation, and using that the contact force moves forward, $f(x,t) = f(t)\delta(x - vt)$.

$$y_r(t_0) = \sum_{k=0}^{K-1} g(x_0 - k\Delta x,x_0; t_0 - k\Delta t,t_0) \times f(t_0 - k\Delta t)\Delta t$$

where $k$ is an integer. Note that the force point, $x = x_0 - k\Delta x$, is expressed relative to the response point, $x_0$. Previous calculations (sec. 1.2) have provided Green's function in the frequency domain, $G_\omega(x,x_0)$, here first made conjugate-symmetric, $G_{\omega - \omega} = G_\omega^\ast$ (the asterisk denoting complex conjugate), to ensure reality and causality, and then transformed to the time domain by a discrete Fourier transform, $g(x,x_0; t,t_0) = \text{DFT}^{-1}[G_\omega(x,x_0)]$. The force, $f(t_0 - k\Delta t)$, known in the past where $k > 0$, must be found by iterative improvements for the very last time step, $k = 0$.The total number of points, $K$, must be so great that the impulse from the most distant point, $x_0 - (K-1)\Delta x$, has decayed to a negligible amplitude before it arrives at the response point, $x_0$.

Wheel

The wheel model is the simplest possible: a rigid mass, $M_w$, acted upon vertically by a constant preload force, $P$, plus the contact force, $f$. Numerically, e.g. with Runge-Kutta, solving the ordinary differential equation $M_w\ddot{y}_w(t) = f(t) - P$ (the dots denoting time derivatives) gives the vertical displacement $y_w$.

Contact

Herzian contact model

The compression of the wheel/rail contact

$$\delta_{\text{lin}} = \gamma_r + y_s - y_w$$
a function of the contact force. Being unknown at the current time step, it can be approximated first with its most recent value, \( f(t_0) \leftarrow f(t_0 - \Delta t) \). According to Hertz, the compression distance is a nonlinear function of the contact force \( f \),

\[
\delta_H = \left[ \frac{2f(1 - \nu)}{G\sqrt{R_a}} \right]^{2/3} \alpha \delta
\]  

(4)

Of course, \( f \geq 0 \) always; \( f = 0 \) implies loss of wheel-rail contact, \( \delta_{lin} \leq 0 \). Here, \( G \) is the shear modulus, \( \nu \) Poisson's ratio and \( R_a \) one of the radii of curvature at the contact point. The function \( \alpha \delta \) depends on the elliptical shape of the contact area. Now, solving

\[ \delta_{lin} - \delta_H(f) = 0 \]

iteratively, yields the Hertzian contact force \( f \). It is then possible to iterate, with the successive force improvements \( f(t_0) \leftarrow f \), until the error \( |f(t_0) - f| < \epsilon \), \( \epsilon \) being a small number. The contact force \( f \) plus rail and wheel deflections, \( y_r \) and \( y_w \), are now known at the current time step.

**State-dependent contact model**

This paragraph follows the preceding, apart from the relation in eq. (4), which is replaced by the state-dependent contact force expression

\[ f(\delta, r) = \begin{cases} k_c(\delta)(r - \tilde{y}_r(r)), & \delta > 0 \\ 0, & \delta \leq 0 \end{cases} \]

where \( k_c(\delta) \) is the state-dependent contact stiffness and \( \tilde{y}_r(r) \) are the state-dependent line texture heights. Definitions and methodology of the state-dependent force formulation, here applied to the wheel-rail contact, is previously presented in [5]. In summary, the widely used Hertzian contact stiffness and the linearized contact filter are replaced by state-dependent contact stiffness and contact filters (varying with wheel-rail relative displacement). Prior to time-domain wheel-rail dynamic modelling, the state-dependent contact stiffness as well as state-dependent contact filtering (in the form of filtered texture heights) are pre-calculated using quasi-static modelling schemes. For this, Boussinesq contact modelling theory is employed, which includes the three-dimensional topography of the wheel and rail within the contact area.

In Figure 2 and Figure 3, examples of computed state-dependent contact stiffness and contact filters are presented for a moderate wheel and rail roughness spectrum (Figure 4).

### 1.3.2 Contact force calculation in the frequency domain

It is shown in [1], similarly as in the TWINS simulation program, that a linear expression (approximation) for the contact force spectrum \( S_{F_0F_0} \) between rail and wheel can be written

\[
S_{F_0F_0} = \frac{1}{\nu} \left[ \Phi_{F_0F_0}(k_0) + \Phi_{w_w}(k_0) \right] \cdot |H(k_0)|^2 \cdot \left[ 1 + k_c(\alpha_r + \alpha_w) \right]^{-2}
\]

(5)
where \( v \) is the train speed, \( \Phi_{F_0F_0}(k_0) \) the rail running surface roughness spectrum, \( \Phi_{W_0W_0}(k_0) \) the wheel running surface roughness spectrum, \( |H(k_0)|^2 \) the contact filter, \( k_c \) the linearized contact spring stiffness, \( \alpha_r \) vertical rail receptance, and \( \alpha_w \) radial wheel receptance. The surface roughness wave number \( k_0 = 2\pi/\lambda_0 \), where \( \lambda_0 \) is the wavelength of the surface roughness.

The force spectrum \( S_{F_0F_0} \) may substitute \( |F|^2 \) in eq. (3), to calculate rail vibration levels and noise radiation.

2. Numerical simulations

2.1 Input data

Rail vibration properties (rail receptance, vibration propagation along the rail, etc), rail/wheel interaction contact force, and rail noise radiation are calculated for a typical ballast track. Different modelling methods are used, to demonstrate when the rail must be modelled discretely supported and a nonlinear contact model between rail and wheel must be used, for a correct description of railway noise generation. Parameters for the track are listed in Table 1. Euler beam theory is used, with a reduction of the moment of inertia to 75 % of its nominal value, thus extending the useful frequency region for vertical rail vibrations up to and above 2 kHz [4].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E )</td>
<td>2.11E11</td>
<td>N/m²</td>
<td>Rail modulus of elasticity</td>
</tr>
<tr>
<td>( \eta_r )</td>
<td>0.02</td>
<td></td>
<td>Rail loss factor</td>
</tr>
<tr>
<td>( I )</td>
<td>28E-6</td>
<td>m⁴</td>
<td>Rail moment of area inertia (75 % of nominal value)</td>
</tr>
<tr>
<td>( m_r )</td>
<td>60</td>
<td>kg/m</td>
<td>Rail mass per unit length</td>
</tr>
<tr>
<td>( M_s )</td>
<td>125</td>
<td>kg</td>
<td>Sleeper mass (half)</td>
</tr>
<tr>
<td>( l )</td>
<td>0.625</td>
<td>m</td>
<td>Sleeper spacing</td>
</tr>
<tr>
<td>( K_p )</td>
<td>100</td>
<td>MN/m</td>
<td>Pad stiffness</td>
</tr>
<tr>
<td>( \eta_p )</td>
<td>0.25</td>
<td></td>
<td>Pad loss factor</td>
</tr>
<tr>
<td>( F_s )</td>
<td>0</td>
<td>N/m-rad</td>
<td>Torsional support stiffness</td>
</tr>
<tr>
<td>( K_b )</td>
<td>100</td>
<td>MN/m</td>
<td>Ballast stiffness (under a half sleeper)</td>
</tr>
<tr>
<td>( \eta_b )</td>
<td>1.0</td>
<td></td>
<td>Ballast loss factor</td>
</tr>
</tbody>
</table>

The combined wheel and rail roughness used to generate line texture heights \( y_i \) is shown as one third octave band wavenumber spectra in Figure 4. The wheel roughness is relatively low and originates from measurements performed on wheels from the Swedish X2-train (trailer coaches) presented in [6]. Also the rail roughness is low since the Swedish intercity track section from which it origins was relatively newly ground when measured. The rail roughness measurements were performed as a part of the EU project QCity and are presented in [7]. The combined roughness levels generated from measurements are extended to cover longer wavelengths (see dashed part of roughness spectra in Figure 4) which extends the investigation to lower excitation frequencies.

![Figure 4: Combined rail-wheel roughness level used as input to dynamic simulations (\( v = 200 \) km/h.](image)

Line texture heights representing the combined wheel and rail roughness were generated by first performing a narrowband FFT on a sequence of uniformly distributed random values (white noise). Secondly, the FFT components are weighted so that the energetic sum of all components lying within the range of each third-octave bandwidth equals the value of the corresponding third-octave band presented in Figure 4. Finally, an inverse FFT result in line texture heights which are used for time-domain prediction of contact forces and wheel-rail vibrations. The above presented measured roughness does not cover the range of short wavelengths required to perform detailed contact modelling with the aim of computing state-dependent parameters. Instead, a set of short wavelength wheel and rail roughness data from [8] is used as reference since the values correspond well with the roughness levels and the decaying slopes presented in [6,7] in a region of wavelengths where the two data sets are overlapping. Using the short wavelength roughness from [8] together with a method to generate surface data from line texture data (see [5]), surface roughness is generated and superimposed the geometrical shape of wheel and rail.

2.2 Point force excitation — rail Green’s function

Radiation of vertical rail vibrations depends on the integral of the Green’s function, \( \int_0^{\infty} |G(x|x_0)|^2 dx \), according to eq. (1). Figure 7 below shows the energy spectrum density (ESD) of rail response for three cases: force point in the middle of a span;
force point on a support position; and mean for all different force positions in a sleeper span.

![Figure 5: Rail receptance. The rail is excited in the middle of sleeper span, and on a support position.](image)

2.2 Contact force calculation

2.3.1 Contact force calculation in the time domain

The contact force is calculated with two different contact models: the nonlinear state-dependent model (sec. 1.3.1) and for comparison, a linearized model using the same iteration scheme. The combined rail/wheel roughness levels according to sec 2.1 and Figure 4 is used. Wheel velocity \( v = 56 \text{ m/s} \) (200 km/h), wheel mass \( M_w = 600 \text{ kg} \), and wheel preload \( P = 67.5 \text{ kN} \). Track parameters according to Table 1 are used. The calculation is done with the periodically supported rail model (sec. 1.2), but also with a continuously supported rail model (see [4]) for comparison, i.e. the rail does not have discrete sleeper supports in this case; instead, the mass and stiffness of the supports is “smeared out” under the rail. The contact force spectra in are shown in Figure 8, and the difference in force spectra with different modellings as compared to the simplest model, continuously supported rail and linear contact model, in Figure 9.

![Figure 6: Propagation of vertical rail vibrations away from the excitation point, in the middle of a sleeper span. The pinned-pinned frequency, 1139 Hz.](image)

![Figure 7: Energy spectrum density (ESD) of vertical rail vibrations.](image)

![Figure 8: Spectrum \( (\Delta f = 1 \text{ Hz}) \) of contact force calculated with two different rail models and two different contact models: (1) continuously supported rail and linear contact model; (2) periodically supported rail and linear contact model; (3) continuously supported rail and nonlinear contact model; (4) periodically supported rail and nonlinear contact model.](image)

![Figure 9: Contact force level difference (reference case: 1. continuous/linear): (1) continuously supported rail and linear contact model; (2) periodically supported rail and linear contact model; (3) continuously supported rail and nonlinear contact model; (4) periodically supported rail and nonlinear contact model.](image)
2.3.2 Contact force calculation in the frequency domain

Equation (5), sec. 1.3.2, yields a (linear) contact force spectrum in the frequency domain. The (combined rail/wheel) roughness spectrum used is shown in Figure 4, rail and wheel receptances in Figure 10. A continuously supported rail model is used here, since effects of varying rail receptance between sleeper supports may strictly not be included in a linear model in the frequency domain (compare receptances in Figure 5). The (linearized) contact spring $k_c = 1$ GN/m, corresponding to a receptance of -180 dB. The calculated contact force spectrum (Figure 11), is corrected by the level difference calculated in the previous section (Figure 9), to account for parametric excitation (periodic rail supports) and nonlinear contact effects.

![Figure 10: Receptances of rail, $\alpha_r$, wheel, $\alpha_w$, and linearized contact spring, $\alpha_c=1/k_c$, used for frequency domain calculation of a contact force spectrum (eq. (5)).](image)

![Figure 11: Calculated contact force spectrum ($\Delta f = 1$ Hz) according to eq. (5), and correction with the contact force level difference between a nonlinear and linear model (Figure 9), to account for parametric excitation and nonlinear effects.](image)

2.3 Rail radiation calculation

Sound radiation is calculated with eq. (3); the force spectrum is corrected to also include effects of parametric excitation and nonlinear effects, according to Figure 11; the energy spectrum density of rail vibrations (including the decay) comes from Figure 7. The radiation ratio of the rail is approximated with the following expression: $\sigma = 2/[1 + (f_c/f)]$, where $f_c = 572$ Hz is the critical frequency of rail radiation. Calculated sound radiation in third-octave bands is shown in Figure 12.

![Figure 12: Total radiated A-weighted sound power (normalized by one hour) due to vertical rail vibrations, for one railway wheel for two cases: including (corrected) and excluding (uncorrected) effects of parametric excitation and nonlinear effects.](image)

3. Discussion

The rail receptance curve (Figure 5) has three maxima/peaks: the first, around 100 Hz occurs when rail and sleeper masses vibrate in-phase on the stiffness of the ballast; the second occurs around 300 Hz when rail and sleepers masses vibrate out-of-phase with the stiffness of the rail pad in between; the third, at 1139 Hz, is the pinned-pinned frequency, where a half bending wavelength equals the distance between the support points, which are nodal points.

The pinned-pinned frequency propagates freely (Figure 6), while vibrations at e.g. 200 Hz decay rapidly. A periodic system such as a periodically supported rail exhibits stop- and pass-band properties, determining the decay rate of vibrations at different frequencies. An advantage of the current model is, that the decay rate, or propagation coefficients, are automatically included in being an inherent property of the Green’s function [1, 2]. Thus, the decay does not have to be measured, because as soon as the rail parameters such as pad stiffness and sleeper spacing (see Table 1) are known, the decay along the rail is also known.

Radiation of vertical rail vibrations depends on the integral of the Green’s function, $\int_{\Omega} |G(x|x_0)|^2 dx$, according to eq. (1). Figure 7 shows energy spectrum density (ESD) of a unit point force excited rail. It is interesting to note the resemblance with the receptance curve (Figure 5). For noise radiation, the averaged curve should be
used, since all parts of the rail in a sleeper span contribute

Below 1000 Hz, the linear and nonlinear contact models yield roughly the same contact force (Figure 8). The periodically supported rail model is required to describe response at the sleeper-passing frequency (Figure 9). Around and above the pinned-pinned frequency, the choice of contact model is crucial. The linear model (with a constant contact stiffness) under predicts the contact force by 5–10 dB as compared to the nonlinear state-dependent model (Figure 9). Contributions to contact force generation due to discrete supports and a nonlinear (state-dependent) contact model is simply the difference between the spectra in Figure 8, and are shown in Figure 9. By adding this level difference to the contact force spectrum calculated with a frequency domain model (eq. (5)), these effects may be included in a noise radiation model (Figure 11).

Sound radiation (Figure 12) peaks around 1000 Hz. Not including parametric excitation and nonlinear effects under predicts total noise radiation from the rail by 2 dB. For other rail/wheel combinations and speeds, the discrepancy may be even greater than so. If also wheel radiation and lateral rail vibration were included in the radiation model, the discrepancy between a linear and a nonlinear model will be greater than the 2 dB resulting from the example used here, since these components radiate efficiently at higher frequencies, where effects of nonlinearities are more pronounced.

4. References


